PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Algebraic Geometrical Methods in Hamiltonian Mechanics [and Discussion]

P. Van Moerbeke, J. T. Stuart and M. Tabor

Phil. Trans. R. Soc. Lond. A 1985 **315**, 379-390
doi: 10.1098/rsta.1985.0045

Phil. Trans. R. Soc. Lond. A **315**, 379–390 (1985)
Printed in Great Britain

Algebraic geometrical methods in Hamiltonian mechanics

By P. van Moerbeke

Department of Mathematics, Brandeis University, Waltham, Massachusetts 02254, *U.S.A., and Department of Mathematics, University of Louvain, B*-1348 *Louvaine-la-Neuve, Belgium*

During the nineteenth century one of the main concerns in mechanics was to solve Hamiltonian systems by quadrature in terms of elliptic and hyperelliptic functions. The vein of research, abandoned for nearly a century, was entirely revived by the recent findings about the Korteweg–de Vries equation. They shed new light and perspective onto problems of finite-dimensional mechanics, which has led to effective and systematic methods for deciding about the complete integrability of Hamiltonian systems. The problems that can be captured by these methods have the common virtue that, when run with complex time, most (complex) trajectories are dense on complex algebraic tori; such a system is called algebraically completely integrable, which is stronger than the customary notion of analytic integrability. Many old and new systems enjoy this property and, in particular, a wide variety of systems that occur in the context of orbits in Kac–Moody Lie algebras. This paper explains how the behaviour at the 'blow-up' time of the solutions to the differential equations enables one to decide about their integrability and also how to derive precise information about the invariant surfaces of the system.

Last century, mechanics was dominated by the question of whether a mechanical problem can be solved by a finite number of algebraic operations which is termed solution 'by quadrature'. This was done, when possible, by finding appropriate variables for which the problem separates; then the system evolves in phase space on low-dimensional manifolds. Proving that a system is integrable by quadrature was very unsystematic and required a great deal of luck. Jacobi (1866) himself was very much aware of this difficulty, and therefore I would like to quote a paragraph from his famous *Vorlesungen über Dynamik*:

> Die Hauptschwierigkeit bei der Integration gegebener Differentialgleichungen scheint in der Einführung der richtigen Variablen zu bestehen, zu deren Auffindung es keine allgemeine Regel giebt. Man muss daher das umgekehrte Verfahren einschlagen und nach erlangter Kenntniss einer merkwürdigen Substitution die Probleme aufsuchen, bei welchen dieselbe mit Glück zu brauchen ist.

Once expressed in the appropriate variables. the inverse of the solution could then be written in terms of integrals of radicals of one or several polynomials; inversion of the latter problem leads to the famous Jacobi–Abel inversion problem of Abelian integrals, which is one of the basic theorems of algebraic geometry. In fact, during most of the last century the developments of mechanics and algebraic geometry were closely intertwined. Within this circle of ideas are the works of Clebsch, Jacobi, Kovalevskaja, Lagrange, Liouville, Neumann, and others. The most prominent problems are the Jacobi geodesic flow on ellipsoids, the integrable cases of the rotation of a solid body around a fixed point, by Euler, Lagrange and Kovalevskaja and the integrable cases of the motion of a solid body in an infinite ideal fluid, by Kirchhoff, Clebsch, Lyapunov, Steklov and others.

Unlike the problems above, which have polynomial invariants in large numbers, one may require that the constants of motion be merely algebraic functions of the coordinates. In this context, I must mention Bruns (1887), who established the absence of new algebraic integrals in the three-body problem in addition to classical integrals (extended by Siegel (1936) to the restricted three-body problem), and Husson (1907), who established that the three known cases of solid bodies mentioned before are the only ones that possess an extra algebraic integral.

In addition to every new generation interpreting the notion of complete integrability in its own way, astronomers and mathematicians have also introduced analytic integrability, meaning that the problem has purely analytic integrals in sufficient number. This ties up with the qualitative behaviour of the trajectories investigated by Poincaré; he pointed out that complete integrability is an exceptional phenomenon and therefore the emphasis was shifted towards non-integrability. Also, a study of the trajectories offers the advantage of coordinate invariance. Non-integrable systems have the properties that the trajectories and, in particular, the separatrices behave in a complicated way: the latter intersect each other transversely, infinitely often. It is then possible, at least in small dimensions, to establish non-integrability of a system very close to an integrable system by showing the appearance of homoclinic intersection points after perturbation. The Melnikov integral method is such a test for low dimensions and leads to establishing the absence of extra analytic integrals for such perturbed systems, independent from the energy. This method, combined with other ideas, has been used recently by Ziglin (1980, 1982, 1983) in the dynamics of the rigid body to show that the only analytically integrable cases among bodies with fixed points are the three known cases.

But the methods just described generally lead to perturbation results and rarely to global ones, i.e. given a large family of Hamiltonian systems, the question is whether one can decide which subfamilies are completely integrable and whether one can find a systematic method to integrate the problem. It now seems that this question can be answered to some extent, within the framework of algebraic integrability in the sense discussed at first. The resolution of the Korteweg–de Vries equation by inverse spectral methods played a crucial role in this development and was at the origin of many new ideas and connections between mechanics, spectral theory, Lie groups and algebraic geometry; they have provided new insights into the old mechanical problems of the last century and many new ones as well. The study of specific systems and equations have led to general schemes, mainly in the realm of Lie algebras, which manufacture lots of completely integrable Hamiltonian systems; some of them can then be recognized to be of genuine mechanical or physical significance. However, given a Hamiltonian system, it often remains hard to fit it into any of these general schemes. All the systems that come from those frameworks are algebraically completely integrable in a precise sense that I will define below. Let me, for the sake of this paper, restrict myself to finite-dimensional systems that are compact as well.

A Hamiltonian equation

$$\dot{z} = f(z) \equiv J(\partial H/\partial z), \quad z \in \mathbb{R}^n,$$

$$J = J(z) = \left\{ \begin{array}{c} \text{skew symmetric matrix with polynomial entries in } z, \\ \text{for which the corresponding Poisson bracket } \{H_i, H_j\} \\ \equiv \langle \partial H_i/\partial z, J\, \partial H_j/\partial z \rangle \text{ satisfies the Jacobi identity,} \end{array} \right\} \tag{1}$$

with polynomial right side will be called algebraically completely integrable (a.c.i.) when:

(i) except for the 'trivial' invariants $H_1, \ldots, H_k$ whose gradients are the null vectors of J, the system possesses $m = \frac{1}{2}(n-k)$ polynomial invariants $H_{k+1}, \ldots, H_{k+m}$ in involution $\{H_i, H_j\} = 0$, having the property that for most values of $c_i \in \mathbb{R}$, the invariant manifolds

$$\bigcap_{i=1}^{k+m} \{H_i = c_i, z \in \mathbb{R}^n\}$$

are compact, connected and, therefore, real tori by the Arnold–Liouville theorem.

(ii) Moreover,

$$\bigcap_{i=1}^{k+m} \{H_i = c_i, z \in \mathbb{C}^n\} \equiv \text{complex algebraic torus } \frac{\mathbb{C}^m}{\text{lattice}} \setminus \mathscr{D},$$

where $\mathscr{D}$ consists of one or several codimension-one subvarieties on the torus; in the natural coordinates $(t_1, \ldots, t_m)$ of these tori, the Hamiltonian flows (run with complex time) $\dot{z} = J\,\partial H_{k+i}/\partial z$ $(i = 1, \ldots, m)$ are straight-line motions and the coordinates $z_i = z_i\ (t_1, \ldots, t_m)$ are meromorphic in $(t_1, \ldots, t_m)$.

Let me now interpret this definition. Since the flow evolves on the m-dimensional tori $T^m = \mathbb{C}^m/\text{lattice}$, the coordinates z_i remain finite on the affine part (i.e. non-compact)

$$\bigcap_{i=1}^{k+m} \{H_i = c_i, z \in \mathbb{C}^n\}$$

of that torus and therefore some or all of the coordinates $z_1, \ldots, z_n$ must blow up along $\mathscr{D}$ in a meromorphic fashion. This is to say that $\mathscr{D}$ is the codimension-one subvariety along which some of the z_i have a pole. By the definition above, the flow (1) is a straight-line motion on T^m; it must therefore intersect the subvariety $\mathscr{D}$ in at least one place (see figure 1) unless the flow remains in the divisor $\mathscr{D}$. Conversely, through every point of $\mathscr{D}$ there is a straight-line motion and therefore a Laurent expansion around that point of intersection. Hence the differential equations must admit Laurent expansions that depend on the $m-1$ parameters defining $\mathscr{D}$ and the $k+m$ constants c_i defining T^m; i.e. they depend on

$$(k+m) + (m-1) = n-1$$

free parameters. These ideas are implicit in Kowalewskaja's (1889*a*, *b*) investigation of the dynamics of the rigid body.

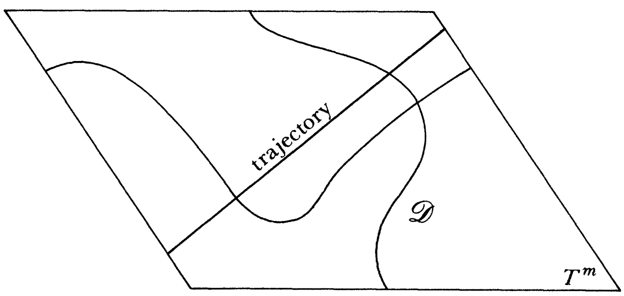

FIGURE 1. The flow on T^m.

THEOREM 1. Necessary condition for algebraic integrability (Adler & van Moerbeke 1982*b*). *If the Hamiltonian flow $\dot{z} = f(z)$ is algebraically completely integrable, then this system of differential equations must admit Laurent expansion solutions in t such that*

(*a*) *each z_i blows up for some value of t, and* (*b*) *at that value of t the Laurent expansion solution z, where*

$$z_i(t) = t^{-k_i}(z_i^{(0)} + z_i^{(1)}t + z_i^{(2)}t^2 + \dots), \quad k_i \in \mathbb{Z} \text{ with some } k_i > 0, \tag{2}$$

admits $n-1$ *free parameters, unless the flow stays within the divisor* $\mathscr{D}$.

Now consider Hamiltonian flows that are (weight)-homogeneous in the following sense:

$$f_i(\alpha^{g_i} z_1, \dots, \alpha^{g_n} z_n) = \alpha^{g_i+1} f_i(z_1, \dots, z_n), \quad \forall \alpha \in \mathbb{C}. \tag{3}$$

For this flow to be algebraically completely integrable, the differential equations (1) must admit Laurent expansion solutions (2), depending on $n-1$ free parameters. For this to occur we must have $k_i = g_i$ and the coefficients in the expansion must satisfy, at the 0th step,

$$f_i(z_1^{(0)}, \dots, z_n^{(0)}) + g_i z_i^{(0)} = 0, \quad i = 1, \dots, n, \tag{4}$$

and at the kth step,

$$(\mathscr{L} - kI)\, z^{(k)} = \text{a polynomial in } z^{(0)}, \dots, z^{(k-1)}, \quad \text{for} \quad k \geqslant 1, \tag{5}$$

where $\mathscr{L}$ is the Jacobian map of (4):

$$\mathscr{L} = \frac{\partial f}{\partial z} + \begin{bmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_n \end{bmatrix}. \tag{6}$$

If $n-1$ free parameters are to appear in the expansion, they must either come from the nonlinear equations (4) or from the eigenvalue problem (5), i.e. $\mathscr{L}$ must have at least $n-1$ integer eigenvalues. These are far fewer conditions than expected, because of the fact that each sufficiently homogeneous constant of the motion *may* lead to an integer eigenvalue of $\mathscr{L}$, which I explain below.

I now proceed to show that a constant of the motion $H(z)$, homogeneous in the sense that for g_i as in (3) and some $k \in \mathbb{Z}$,

$$H(\alpha^{g_i} z_1, \dots, \alpha^{g_n} z_n) = \alpha^k H(z), \quad \forall \alpha \in \mathbb{C}, \tag{7}$$

implies that k is an eigenvalue of $\mathscr{L}$ unless $\partial H/\partial z$ vanishes for every solution of (4). To see this observe that the vector $z^0(t) \equiv (z_1^0\, t^{-g_1}, \dots, z_n^0\, t^{-g_n})$ with $z_i^0(t)$ satisfying (4) is a special solution of $\dot{z} = f(z)$. Then $\zeta \equiv (\zeta_1^0\, t^{l-g_1}, \dots, \zeta_n^0\, t^{l-g_n})$ is a solution of the variational equation

$$\dot{\zeta} = \left\langle \frac{\partial f}{\partial z}\bigg|_{z=z^0(t)}, \zeta \right\rangle$$

around the special solution $z^0(t)$ with the vector ζ^0 being an eigenvector of $\mathscr{L}$ with eigenvalue l. If $H(z)$ is a constant of the motion for $\dot{z} = f(z)$, then $\langle \partial H/\partial z, \zeta \rangle$ is a constant for any solution of the variational equation, which implies that $l = k$ must hold unless this expression vanishes for every solution of (4). To conclude, the homogeneity k of the constant H must be an eigenvalue of $\mathscr{L}$ (see Yoshida 1983).

The criterion of theorem 1 together with this conclusion can effectively be used to pin down, among a family of Hamiltonian systems (depending, say, on various parameters), those that are algebraically completely integrable. Although the sufficiency of the criterion of theorem 1 remains to be shown in full generality, I shall indicate how it might work in practice. If the mechanical motion is algebraically completely integrable, then, according to the criterion, the operator $\mathscr{L}$ must have $n-1$ integer eigenvalues; as mentioned above, these are fewer conditions than expected, because of the fact that the known constants of the motion (for example, the energy and the trivial constants) may already lead to integer eigenvalues of $\mathscr{L}$. Once such a subfamily of Hamiltonian systems has been found, one must show that they are algebraically completely integrable. The argument can be broken up according to the following four steps.

(i) One must show the existence of the Laurent expansions, which requires an argument precisely every time k is an integer eigenvalue of $\mathscr{L}$, and therefore $\mathscr{L}-kI$ is not invertible (see equation (5)).

(ii) One shows the existence of the remaining constants of the motion in involution so as to reach the number $m+k$.

(iii) For given $c_1, \ldots, c_n$, the set

$$\mathscr{D} \equiv \{\text{the Laurent solutions } z(t) = (t^{-g_i}(z_i^{(0)}+z_i^{(1)}t+\ldots))_{1\leqslant i\leqslant n} \text{ such that } H_j(z(t)) = c_j\}$$

defines one or several $(m-1)$-dimensional algebraic varieties (divisor), having the property that

$$\bigcap_{i=1}^{k+m} \{H_i = c_i,\, z\in\mathbb{C}^n\} \cup \mathscr{D}$$

is a smooth compact, connected variety with m commuting vector fields independent at every point, that is a complex algebraic torus $T^m = \mathbb{C}^m/\text{lattice}$. The flows $J(\partial H_{k+1}/\partial z), \ldots, J(\partial H_{k+m}/\partial z)$ are straight-line motions on T^m.

(iv) A great deal of information can be obtained about the periods and the action-angle variables from the divisor $\mathscr{D}$.

Next I explain how these ideas can be used on an example. Consider the motion of a solid in an unbounded, irrotational, incompressible fluid (considered last century by G. R. Kirchhoff); they have the form

$$\left.\begin{aligned} \dot{p} &= p\times\partial H/\partial l, \\ \dot{l} &= p\times\partial H/\partial p + l\times\partial H/\partial l, \end{aligned}\right\} \tag{8}$$

where $p\in\mathbb{R}^3$ is the velocity of a point fixed relative to the solid and $l\in\mathbb{R}^3$ is the angular velocity of the body expressed with respect to a frame of reference also fixed relative to the solid. Also, H is a quadratic polynomial in p and l, which accounts for the kinetic energy of the solid and the liquid. This motion can be regarded as a geodesic motion on the group of rigid motions $E_3 = \mathrm{SO}(3)\times\mathbb{R}^3$ with a left-invariant metric, or by reduction, as a Hamiltonian flow on the coadjoint orbits of its dual Lie algebra, identified with $\mathrm{so}(3)\times\mathbb{R}^3$. Hence the motion has the trivial coadjoint orbit invariants $\langle p,p\rangle$ and $\langle p,l\rangle$. It transpires that this is a special case of a more general system of equations written as

$$\left.\begin{aligned} \dot{x}' &= x'\times\partial H/\partial x' + x''\times\partial H/\partial x'', \\ \dot{x}'' &= x''\times\partial H/\partial x' + x'\times\partial H/\partial x'', \end{aligned}\right\} \tag{9}$$

where $x' = (x_1, x_2, x_3)$ and $x'' = (x_4, x_5, x_6)$. The first set can be obtained from the second by putting

$$(x', x'') = (l, p/\epsilon) \tag{10}$$

and letting $\epsilon \to 0$. The latter set of equations is the geodesic flow on the group SO(4) for a left-invariant metric defined by the quadratic form H, which can be written in short as

$$\dot{X} = [X, \partial H/\partial X], \quad X \in \mathrm{so}(4), \tag{11}$$

where
$$X = \begin{bmatrix} 0 & -x_3 & x_2 & -x_4 \\ x_3 & 0 & -x_1 & -x_5 \\ -x_2 & x_1 & 0 & -x_6 \\ x_4 & x_5 & x_6 & 0 \end{bmatrix} = (X_{ij})_{1 \leqslant i, j \leqslant 4}. \tag{12}$$

Often it is more convenient to use the coordinates $z' = (z_1, z_2, z_3)$ and $z'' = (z_4, z_5, z_6)$ with $z_i = x_i + x_{i+3}$ and $z_{i+3} = x_i - x_{i-3}$ $(1 \leqslant i \leqslant 3)$; they correspond to the decomposition of $\mathrm{so}(4) \simeq \mathrm{so}(3) \oplus \mathrm{so}(3)$. In these coordinates the equations become $\dot{z} = f(z)$, i.e.

$$\dot{z}' = z' \times \partial H/\partial z', \quad \dot{z}'' = z'' \times \partial H/\partial z''; \tag{13}$$

the quadratic form H depends on 21 parameters, which by conjugation can be reduced to 15. For present purposes let H take the simpler form

$$H(z) = \frac{1}{2} \sum_{i=1}^{6} \lambda_i z_i^2 + \sum_{i=1}^{3} \lambda_{i,i+3} z_i z_{i+3} \tag{14}$$

with the non-degeneracy assumption $(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_6)(\lambda_5 - \lambda_4)(\lambda_6 - \lambda_5)$ $\lambda_{14}\lambda_{25}\lambda_{36} \neq 0$. The equations have, in addition to the energy $Q_3 = H$, two trivial constants of the motion

$$Q_1 = z_1^2 + z_2^2 + z_3^2 \quad \text{and} \quad Q_2 = z_4^2 + z_5^2 + z_6^2.$$

The Laurent solutions (2) have $k_i = 1$ $(1 \leqslant i \leqslant n)$ and have, therefore, simple poles; the three constants of the motion Q_1, Q_2 and Q_3 are quadrics, so that 2 is a triple eigenvalue of $\mathscr{L}$, as follows from the general argument above. Moreover, the quadratic nature of the differential equations $\dot{z} = f(z)$ implies that -1 is an eigenvalue of $\mathscr{L}$. The fact that $g_i = k_i = 1$ in (6) implies $\mathrm{Tr}\, \mathscr{L} = 6$; hence the two remaining eigenvalues (positive or zero) must add up to 1. Therefore, one of the eigenvalues of $\mathscr{L}$ must be 0 (another then being, automatically, 1); this is achieved only when the set of six nonlinear homogeneous quadratic equations

$$z^{(0)} + f(z^{(0)}) = 0 \tag{15}$$

governing the leading term $z^{(0)}$ of the expansion admits a curve of solutions rather than a discrete set of points, as one would expect from a dimension count. Using arguments based on the principles (i) to (iv), one concludes that the geodesic flow on SO(4) is algebraically completely integrable if and only if the equations (15) have a curve in common; this occurs precisely in the three cases described in the next theorem.

Theorem 2 (Adler & van Moerbeke 1984). *The geodesic flow on* SO(4) *for the metric defined by the non-degenerate quadratic form* (14) *is algebraically completely integrable if and only if*

(1) *The quadratic form H is diagonal with respect to the coordinates* (12) *of* SO(4), *i.e.*

$$2H = \sum_{1 \leqslant i < j \leqslant 4} \Lambda_{ij} X_{ij}^2 \quad \textit{with} \quad \Lambda_{ij} = (\beta_i - \beta_j)/(\alpha_i - \alpha_j), \quad \beta_i, \alpha_i \in \mathbb{C} \quad \textit{(Manakov's (1976) metric).}$$

Then the extra-invariant Q_4 is quadratic and the flow is a straight-line motion on two-dimensional complex algebraic tori $\mathbb{C}^2$/lattice (Abelian variety), where the lattice is generated by the four vectors

$$\begin{bmatrix} 2 & 0 & a & c \\ 0 & 4 & c & b \end{bmatrix} \text{ for appropriately chosen } a, b, c \in \mathbb{C}, \text{ such that } \mathrm{Im} \begin{bmatrix} a & c \\ c & b \end{bmatrix} > 0.$$

More precisely

$$\bigcap_{i=1}^{4} \{Q_i = c_i,\ z \in \mathbb{C}^6\} = Prym(\mathscr{C}/\mathscr{C}_0)\backslash(\mathscr{D} = \text{a curve of genus } 9),$$

where the elliptic curve $\mathscr{C}_0$ is defined as

$$\mathscr{C}_0 = \{(t_1, t_2, t_3, t_4) \in \mathbb{P}^3(\mathbb{C}) \text{ such that } \Sigma t_i Q_i \text{ has rank 3, i.e. a sum of 3 squares}\} \quad (16)$$

and $\mathscr{C}$ is a double cover of $\mathscr{C}_0$ ramified at the four intersection points of the line $\Sigma_1^4 t_i c_i = 0$ with $\mathscr{C}_0$ (figure 2). A *Prym* variety is an Abelian variety (complex algebraic torus) constructed from

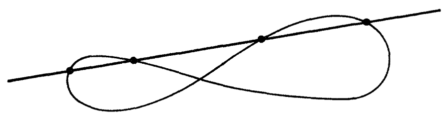

FIGURE 2. The four intersection points of the line $\Sigma_1^4 t_i c_i = 0$ with $\mathscr{C}_0$.

a double cover $\mathscr{C}$ of a curve $\mathscr{C}_0$: if τ is the involution on $\mathscr{C}$ that interchanges the sheets of $\mathscr{C}$, then τ extends by linearity to a map $\tau: Jac(\mathscr{C}) \to Jac(\mathscr{C})$, where $Jac(\mathscr{C})$ is the Abelian variety defined by the periods of the curve $\mathscr{C}$. Up to points of order two, $Jac(\mathscr{C})$ splits up into an even part $Jac(\mathscr{C}_0)$ and an odd part $Prym(\mathscr{C}/\mathscr{C}_0)$. The periods of this *Prym* variety provide the exact periods of the motion in terms of (explicit) Abelian integrals; see Haine (1983, 1984) and the Appendix to Adler & van Moerbeke (1982*b*) by D. Mumford. Observe that the requirement on the form of A_{ij} is equivalent to the identity

$$[X, \beta] + [\alpha, \partial H/\partial X] = 0$$

for diagonal matrices $\alpha = \operatorname{diag}(\alpha_1, \ldots, \alpha_4)$ and $\beta = \operatorname{diag}(\beta_1, \ldots, \beta_4)$. That is to say that the flow (11) can be rewritten as

$$(X + \alpha h)^{\cdot} = [X + \alpha h, \partial H/\partial X + \beta h], \quad X, \partial H/\partial X \in \mathrm{so}(n) \quad (17)$$

with an indeterminate h. For the sake of this discussion choose n to be arbitrary. This is a flow in the dual of the Kac–Moody Lie algebra

$$\mathscr{L} = \tilde{\mathrm{gl}}(n, \mathbb{R}) = \left\{ A = \sum_{-\infty}^{M} A_i h^i,\ A_i \in \mathrm{gl}(n, \mathbb{R}),\ M \text{ arbitrary} \right\}$$

for the somewhat unusual pairing

$$\langle A, B \rangle_1 = \sum_{i+j=-1} \operatorname{tr}(A_i B_j).$$

The Lie algebra $\mathscr{L}$ has a natural decomposition

$$\mathscr{L} = \mathscr{L}_{-\infty, -1} + \mathscr{L}_{0, \infty},$$

where
$$\mathscr{L}_{ij} = \left\{ \sum_{k:i}^{j} A_k h^k,\ A_k \in \mathrm{gl}(n, \mathbb{R}) \right\};$$

observe that $\mathscr{L}^{\perp}_{-\infty, -1} = \mathscr{L}_{-\infty, -1}$ and $\mathscr{L}^{\perp}_{0, \infty} = \mathscr{L}_{0, \infty}$. The group underlying $\mathscr{L}_{-\infty, -1}$ acts coadjointly onto $\mathscr{L}_{0, \infty}$ according to the customary rule of conjugation, followed by registering the non-negative powers of h only. The orbits described in this way come equipped with a symplectic structure. All the functions invariant under the action of $\mathscr{L}$ itself commute along those orbits; in particular, the flow (17) evolves on the coadjoint orbit through $X+\alpha h \in \mathscr{L}_{0,\infty}$ and $\partial H/\partial X+\beta h$ is the gradient of a function invariant under $\mathscr{L}$. Therefore, by the theorem of Adler (1979), Kostant (1979) and Symes (1980), this flow has plenty of constants in involution, and by a theorem of Adler & van Moerbeke (1980*a*, *b*) the flow linearizes on the Abelian variety defined by the periods of the curve $\mathscr{K}$ (in short: $Jac(\mathscr{K})$) defined by

$$Q(h,z) \equiv \det(X+\alpha h - zI) = 0$$

and, to be more specific, on $Prym(\mathscr{K}/\mathscr{K}_0)$, where $\mathscr{K}_0$ is the curve obtained from $\mathscr{K}$ by identifying $(z,h) \to (-z,-h)$. It is interesting to observe that the tori obtained by the asymptotic methods and those built from the orbits in the Kac–Moody Lie algebra are *not identical* but only isogeneous (i.e. obtained by doubling some periods), as shown by Haine (1983).

Going back to the X_i coordinate and taking an appropriate limit when $\epsilon \to 0$, leads to the Hamiltonian

$$H = \tfrac{1}{2}(c_1 l_1^2 + c_2 l_2^2 + c_3 l_3^2 + b_1 p_1^2 + b_2 p_2^2 + b_3 p_3^2)$$

with c_i and b_i satisfying

$$c_2 c_3 (b_3 - b_2) + c_3 c_1 (b_1 - b_3) + c_1 c_2 (b_2 - b_1) = 0.$$

This is the case of the rigid body in a fluid, which was integrated by Clebsch.

(2) *The quadratic form*

$$H = \frac{1}{2}\sum_1^6 \lambda_i z_i^2 + \sum_1^3 \lambda_{i,i+3} z_i z_{i+3} \tag{18}$$

satisfies (let $\Delta_{ij} \equiv \lambda_i - \lambda_j$*)*

$$(\lambda_{14}^2, \lambda_{25}^2, \lambda_{36}^2)(\Delta_{12}\Delta_{46} - \Delta_{13}\Delta_{45})^2 = \Delta_{21}\Delta_{54}\Delta_{32}\Delta_{65}\Delta_{13}\Delta_{46}\left[\frac{(\Delta_{25}-\Delta_{36})^2}{\Delta_{32}\Delta_{65}}, \frac{(\Delta_{36}-\Delta_{14})^2}{\Delta_{13}\Delta_{46}}, \frac{(\Delta_{14}-\Delta_{25})^2}{\Delta_{21}\Delta_{54}}\right]$$

with the product $\lambda_{14}\lambda_{25}\lambda_{36}$ *being rational in* $\lambda_1, \ldots, \lambda_6$.

Then the extra invariant Q_4 is quadratic and the flow linearizes on two-dimensional hyperelliptic Jacobians. More precisely,

$$\bigcap_1^4 \{Q_i = c_i, z \in \mathbb{C}^6\} = Jac(\mathscr{C}) \backslash \mathscr{D}$$

where $\mathscr{D}$ represents a divisor of genus 17, which contains 4 translates of the Θ-divisor in $Jac(\mathscr{C})$, each of which is isomorphic to $\mathscr{C}$, and where the hyperelliptic curve $\mathscr{C}$ is again a double cover of the curve $\mathscr{C}_0$ defined by (16); in this case $\mathscr{C}_0$ is isomorphic to $\mathbb{P}^1$. The periods of the motion are given by the periods of the hyperelliptic curve $\mathscr{C}$. For this metric, it is interesting to point out that, up to now, the Kac–Moody Lie algebra interpretation of this motion is unknown; therefore the method of Laurent expansions is the only one available for this case.

Now let us have a closer look at the geometry of the Laurent solutions and the divisor $\mathscr{D}$. Make an appropriate linear change of variables $z \to y$ and consider a new basis $Q_1', \ldots, Q_4'$ for the linear span of quadrics Q_1, Q_2, Q_3, Q_4:

$$Q_1' = y_2^2 - y_3^2, \quad Q_2' = 2y_1^2 + y_6^2, \quad Q_3' = 2y_4^2 + y_5^2,$$
$$Q_4' = (y_1 - y_4)^2 + (y_2 - y_5)^2 + (y_3 - y_6)^2.$$

Then the affine variety $\bigcap_1^4 \{Q'_i = c_i, z \in \mathbb{C}^6\}$ comes equipped with two commuting Hamiltonian vector fields of a simple kind, one of which has the form

$$\left.\begin{aligned} \dot{y}_1 &= y_2 y_6, \quad \dot{y}_2 = -y_3(y_1+y_4), \quad \dot{y}_3 = -y_2(y_1+y_4), \\ \dot{y}_4 &= y_5 y_3, \quad \dot{y}_5 = -2y_4 y_3, \quad \dot{y}_6 = -2y_1 y_2. \end{aligned}\right\} \tag{19}$$

One verifies that the invariant manifold embedded into projective space

$$I \equiv \bigcap_{i=1}^{4} \{Q'_i = c_i y_0^2, y \in \mathbb{P}^6\}$$

intersects the hyperplane at infinity $y_0 = 0$ according to eight curves isomorphic to $\mathbb{P}^1$, four of which $(L_1, \ldots, L_4)$ are simple covers, and four of which $(L'_1, \ldots, L'_4)$ are double covers, of $\mathbb{P}^1$; the variety I is singular (normal crossings) along $L'_1, \ldots, L'_4$ and smooth along $L_1, \ldots, L_4$ (figure 3). The solutions to the system of differential equations (19) intersect L'_i transversely, and at each point of L'_i the vector fields are doubly ambiguous; however, the solutions to (19) totally ignore the lines L_i and the same facts hold for the other vector field commuting with the first.

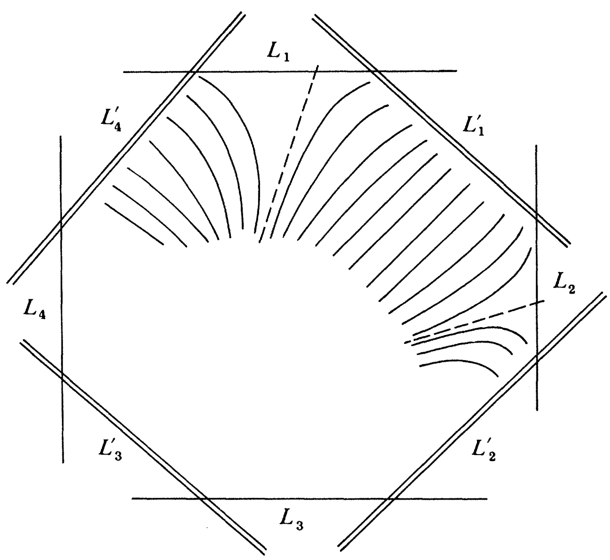

FIGURE 3. The variety I.

To regularize (i.e. make the flow parallel) the flow at infinity, one must blow down the variety along the lines L_i and separate the two sheets of I along L'_i. The new complex surface obtained in this fashion is compact, complex and smooth and has two commuting vector fields on it; it is therefore a complex algebraic torus (Abelian surface) and the eight curves at infinity turn into four hyperelliptic curves, intersecting triply into four points.

Finally, taking the fluid limit $\epsilon \to 0$ yields

$$H = \tfrac{1}{2}\sum a_i[l_i - (a_1+a_2+a_3-a_i)\,p_i]^2,$$

which is the Lyapunov–Steklov case of rigid motion in fluids.

(3) *The quadratic form H can be reduced to the simple form*

$$H = 2(y_1 y_4 + y_2 y_5 - y_3 y_6) - (3s-1)(y_4^2 - y_2 y_5) + (3s+1)(y_5^2 - y_1 y_4)$$

after a linear change of variables from z to y and after replacing H by an appropriate linear combination of Q_1, Q_2 and H.

This is legitimate because it does not change the corresponding Hamiltonian vector field. Then the differential equations take on the form

$$\dot{y}_1 = y_3 y_5, \qquad \dot{y}_4 = \frac{2s}{3s-1} y_5 y_6 + \frac{s-1}{3s-1} y_2 y_3,$$

$$\dot{y}_2 = y_4 y_6, \qquad \dot{y}_5 = \frac{s+1}{3s+1} y_1 y_6 + \frac{2s}{3s+1} y_3 y_4,$$

$$\dot{y}_3 = -\frac{s-1}{2} y_4 y_5 + y_1 y_5 + \frac{s+1}{2} y_1 y_2, \qquad \dot{y}_6 = \frac{s+1}{2} y_4 y_5 + y_2 y_4 - \frac{s-1}{2} y_1 y_2.$$

This Hamiltonian flow has a quartic invariant Q_4, in addition to the three quadratic invariants Q_1, Q_2 and $Q_3 = H$ and

$$\bigcap_1^4 \{Q_i = c_i, z \in \mathbb{C}^6\}$$

becomes a complex algebraic torus after completion with a divisor $\mathscr{D}$, which is a curve of genus 25 intersecting itself at 8 points. The smooth model (i.e. after blowing up the intersection points) of this curve has genus 17 and is a fourfold unramified cover of a curve $\mathscr{C}$ of genus 5. The latter is a ramified cover of a hyperelliptic curve $\mathscr{C}_0$ of genus 2; in fact, all the dynamics takes place on *Prym* $(\mathscr{C}/\mathscr{C}_0)$, which provides the periods of the motion and on which the flow linearizes. For this metric, the Kac–Moody interpretation is missing as well.

To conclude, the method of Laurent expansion enables one to find, among a family of Hamiltonian systems, those that are algebraically completely integrable. The expansions can then be used to manufacture the tori, without ever going through the delicate procedure of blowing up and down. Information about the tori can then be gathered from the divisor $\mathscr{D}$.

The support of a National Science Foundation grant no. 8102696 is gratefully acknowledged.

References

Adler, M. 1979 On a trace functional for pseudo-differential operators and the symplectic structure of the Korteweg–de Vries equation. *Invent. Math.* **50**, 219–248.

Adler, M. & van Moerbeke, P. 1980*a* Completely integrable systems, Euclidean Lie algebras and curves. *Adv. Math.* **38**, 267–317.

Adler, M. & van Moerbeke, P. 1980*b* Linearization of Hamiltonian systems, Jacobi varieties and representation theory. *Adv. Math.* **38**, 318–379.

Adler, M. & van Moerbeke, P. 1982*a* Kowalewski's asymptotic method, Kac–Moody Lie algebras and regularizaton. *Communs math. Phys.* **83**, 83–106.

Adler, M. & van Moerbeke, P. 1982*b* The algebraic integrability of geodesic flow on SO(4). *Invent. Math.* **67**, 297–326.

Adler, M. & van Moerbeke, P. 1984 Geodesic flow on SO(4) and the intersection of quadrics. *Proc. natn Acad. Sci. U.S.A.* **81**, 4613–4616.

Adler, M. & van Moerbeke, P. 1984 Algebraic geometrical methods in Hamiltonian mechanics. In *Progress in mathematics*. Birkhaüser.

Bruns, H. 1887 Über die Integrale des Vielkörper-Problems. *Acta math., Stockh.* **11**, 25–96.
Clebsch, A. 1871 Die Bewegung eines starren Körpers in einer Flüssigkeit. *Math. Annln* **3**, 238–268.
Haine, L. 1983 Geodesic flow on SO(4) and Abelian surfaces. *Math. Annln* **263**, 435–472.
Haine, L. 1984 The algebraic complete integrability of geodesic flow on SO(N). *Communs math. Phys.* **94**. 271–287.
Husson, Ed. 1907 Sur un théorème de M. Poincaré, relativement an mouvement d'un solide pesant. *Acta math., Stockh.* **31**, 71–88.
Jacobi, C. G. J. 1866 Vorlesungen über Dynamik (Königsberg). In *gesammelte werke. Supplementband, Berlin (1884)*. Sechsundzwanzigste Vorlesung (Elliptische Coordinaten).
Kostant, B. 1979 The solution to a generalized Toda Lattice and representation theory. *Adv. math.* **34**, 195–338.
Kowalewsky, S. 1889*a* Sur le problème de la rotation d'un corps solide autour d'un point fixe. *Acta Math., Stockh.* **12**, 177–232.
Kowalewsky, S. 1889*b* Sur une propriété du système d'équations différentielles qui définit la rotation d'un corps solide autour d'un point fixe. *Acta Math., Stockh.* **14**, 81–93.
Lyapunov, A. M. 1893 *Annals of Kharkov. Math. Soc. Ser.* 2, **4**, no. 1–2, 81–85; *Gesammelte Werke*, vol. 1, 320–324.
Manakov, S. V. 1976 Remarks on the integrals of the Euler equations of the *n*-dimensional heavy top. *Funct. Anal. Appl.* **10**, 93–94.
van Moerbeke, P. 1983 Algebraic complete integrability of Hamiltonian systems and Kac–Moody Lie algebras. *Proc. Int. Congress of Math., Warszawa, August 1983.*
Siegel, C. L. 1936 Über die algebraischen Integrale des restringierten Dreikörper problems. *Trans. Am. math. Soc.* **39**, 225–233.
Symes, W. 1980 Systems of Toda type, inverse spectral problems and representation theory. *Invent. Math.* **59**, 13–53.
Yoshida, H. 1983 Necessary conditions for the existence of algebraic first integrals. I. Kowalewski's exponents. II. Conditions for algebraic integrability. *Celestial Mech.* **31**, 363–379, 381–399.
Ziglin, S. L. 1980 Splitting of separatrices, branching of solutions and non-existence of an integral in the dynamics of a solid body. *Trudў mosk. mat. Obshch.* **41**, 287–303.
Ziglin, S. L. 1982 Branching of solutions and non-existence of first integrals in Hamiltonian mechanics. *Funkt. Anal. Appl.* **16**(3), 30–41. (Transl: *Funct. Anal. Appl.* **16**, 181–189 (1982).
Ziglin, S. L. 1983 Branching of solutions and the non-existence of first integrals in Hamiltonian mechanics. II. *Funkt Anal. Appl.* **17**(1), 8–23. (Transl: *Funct. Anal. Appl.* **17** (1), 6–17.)

Discussion

J. T. Stuart, F.R.S. (*Department of Mathematics, Imperial College, London, U.K.*). It seems strange that the use of Kac–Moody algebras may not give the tori correctly, but perhaps with period doubling, in contrast with the statement that the correct tori would be obtained by asymptotics. This indicates a need for caution in interpretation of the result for tori calculated from use of Kac–Moody algebras.

P. van Moerbeke. A striking example of this phenomenon already appears in Jacobi's geodesic flow problem on ellipsoids. The natural coordinates are not meromorphic on the tori but only their squares, if one linearizes the problem via Kac–Moody Lie algebras. In contrast these natural coordinates are meromorphic on the actual invariant surfaces and on their natural completion into tori by using the Laurent expansions. The relation between the two sets of tori is as follows: one set can be obtained from the other by doubling some but not all periods.

M. Tabor (*Department of Applied Physics, Columbia University, New York, U.S.A.*). Can algebraically integrable systems admit movable (i.e. positions dependent on initial conditions) essential singularities or rational branch points? If not, for what classes of integrable system would such singularities be allowed?

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

P. van Moerbeke. Theorem 1 implies that for almost all initial conditions the solution to an algebraically completely integrable system blows up in a meromorphic way after some finite (complex) time; therefore, if a solution was to develop an essential singularity at all, it would be so for a very thin set of initial conditions; it is strongly suspected that even this does not happen. In the definition of algebraic complete integrability I should point out the importance of the requirement that the coordinates be meromorphic functions on the tori. This excludes all bad behaviour of the solutions.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES